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CHANGE OF BASE FOR LOCALLY INTERNAL CATEGORIES

Renato BETTI
Università di Torino, Dipartimento di Matematica, via Principe Amedeo 8, 10123 Torino, Italy

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Locally internal categories over an elementary topos $\mathscr{F}$ are regarded as categories enriched in the bicategory Span$\mathscr{F}$. The change of base is considered with respect to a geometric morphism $\mathscr{F} \to \mathscr{E}$. Cocompleteness is preserved, and the topos $\mathscr{F}$ can be regarded as a cocomplete, locally internal category over $\mathscr{E}$. This allows one in particular to prove an analogue of Diaconescu's theorem in terms of general properties of categories.

Introduction

Locally internal categories over a topos $\mathscr{E}$ can be regarded as categories enriched in the bicategory Span$\mathscr{E}$ and in many cases their properties can be usefully dealt with in terms of standard notions of enriched category theory. From this point of view Betti and Walters [2, 3] study completeness, ends, functor categories, and prove an adjoint functor theorem.

Here we consider a change of the base topos, i.e. a geometric morphism $p : \mathscr{F} \to \mathscr{E}$. In particular $\mathscr{F}$ itself can be regarded as locally internal over $\mathscr{E}$. Again, properties of p can be expressed by the enrichment (both in Span$\mathscr{F}$ and in Span$\mathscr{E}$) and the relevant module calculus of enriched category theory applies directly to most calculations.

Our notion of locally internal category is equivalent to that given by Lawvere [6] under the name of *large category with an $\mathscr{E}$-atlas*, and to Benabou's *locally small fibrations* [1]. It can be described in terms of suitable families of $\mathscr{E}/u$ enriched categories (u varying in $\mathscr{E}$), as in [8], or in terms of *indexed categories*, as in [7].

In Section 1, we recall the main notions and fix the terminology relative to categories enriched in bicategories of the type Span$\mathscr{E}$. Such notions extend the usual ones of the V-enriched case (for which our reference is [5]).

1. Locally internal categories

A *locally internal category* over $\mathscr{E}$ is a (Span$\mathscr{E}$)-category $\mathscr{X}$ with substitution along maps. A *map* f is an arrow of $\mathscr{E}$ regarded in Span$\mathscr{E}$ (such arrows are

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characterized by having a right adjoint f°) and the definition means that, for any object x over u and any map $f: v \to u$, the composite module

$$\mathscr{X}(x,-)\cdot f : v \to u \dashrightarrow \mathscr{X}$$

is representable:

$$\mathscr{X}(x,-)\cdot f \cong \mathscr{X}(f^*x,-).$$

Equivalently $f^\circ \cdot \mathscr{X}(-,x) \cong \mathscr{X}(-,f^*x)$ holds.

Functors of (Span $\mathscr{E}$)-categories preserve substitutions, hence we denote by Loc($\mathscr{E}$) the 2-category of locally internal categories, all their functors and all natural transformations between them.

In particular one-object categories (i.e. monads in Span $\mathscr{E}$) are internal categories, and *monad maps* are internal functors.

An internal category $\mathbf{C} = (C_0 \xleftarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_0)$ becomes a locally internal one, in a universal way, by the *externalization process* $\mathscr{L}$: objects of $\mathscr{L}\mathbf{C}$ over u are maps $u \to C_0$, homs are given by

$$\mathscr{L}\mathbf{C}(f,g) = g^\circ \cdot \delta_1 \cdot \delta_0^\circ \cdot f : u \to C_0 \dashrightarrow C_0 \to v.$$

Substitutions are given by the compositions of maps, and the above assignment extends to a 2-functor

$$\mathrm{Cat}(\mathscr{E}) \xrightarrow{\mathscr{L}} \mathrm{Loc}(\mathscr{E}).$$

For locally internal categories, we consider limits and colimits of functors $F: \mathbf{A} \to \mathscr{X}$ when $\mathbf{A}$ is internal. Precisely the colimit $\alpha * F$ of F indexed by the module $\alpha : \mathbf{A} \dashrightarrow u$ is an object, if it exists, which represents the right Kan extension of α through the module $\mathscr{X}(F,-)$:

$$\mathscr{X}(\alpha * F, -) \cong \hom_A(\alpha, \mathscr{X}(F,-)) : u \dashrightarrow \mathscr{X}.$$

Analogously, the limit $\{\beta, F\}$ indexed by the module $\beta : u \dashrightarrow \mathbf{A}$ is an object which represents the right lifting:

$$\mathscr{X}(-,\{\beta,F\}) \cong \hom^A(\beta, \mathscr{X}(-,F)) : \mathscr{X} \dashrightarrow u. \tag{1}$$

From an internal category $\mathbf{C}$, a locally internal one, $\mathscr{P}\mathbf{C}$, can be obtained in order to classify modules, in the sense that there is an isomorphism of categories

$$\mathrm{Mod}(\mathbf{C}, \mathscr{X})^{\mathrm{op}} \cong \mathrm{Fun}(\mathscr{X}, \mathscr{P}\mathbf{C}) \tag{2}$$

natural in $\mathbf{C}$. Without ambiguity, we can use the same name for objects corresponding under the bijection of (2).

In particular the Yoneda embedding $\mathbf{C} \to \mathscr{P}\mathbf{C}$ corresponds to the identity module $\mathbf{C} \dashrightarrow \mathbf{C}$, under the isomorphism (2).

The objects over u of the category $\mathscr{P}\mathbf{C}$ are the modules $\mathbf{C} \dashrightarrow u$ and homs are given by right Kan extensions. Observe that the objects over the terminal object 1 are internal presheaves, so the fiber of $\mathscr{P}\mathbf{C}$ over u can be seen as the category of

u-indexed families of the topos $\mathscr{E}^{C^{\mathrm{op}}}$. In particular $\mathscr{P}1$ is $\mathscr{E}$ regarded as locally internal over itself.

For any internal category $\mathbf{C}$, the (Span $\mathscr{E}$)-category $\mathscr{P}\mathbf{C}$ is locally internal, the substitution $f^*\alpha$ of $\alpha : \mathbf{C} \to\!\!\!\!\!+ u$ along the map $f: Y \to u$ being the module $f^\circ \cdot \alpha : \mathbf{C} \to\!\!\!\!\!+ v$. Moreover, $\mathscr{P}\mathbf{C}$ is complete and cocomplete: with the above notation, the limit $\{\beta, F\}$ is given by the right lifting $\hom^u(\beta, F)$, the colimit $\alpha * F$ by the composite module $\alpha \cdot F : \mathbf{C} \to\!\!\!\!\!+ u$.

In fact, $\mathscr{P}\mathbf{C}$ is the free cocompletion of $\mathbf{C}$, in the sense that any functor $F: \mathbf{C} \to \mathscr{X}$ (where $\mathscr{X}$ is a cocomplete category) factors uniquely, up to isomorphism, through the Yoneda embedding $\mathbf{C} \to \mathscr{P}\mathbf{C}$ followed by a cocontinuous functor $L_F : \mathscr{P}\mathbf{C} \to \mathscr{X}$. The functor L_F is given by the colimit:

$$L_F(\alpha) = \alpha * F,$$

hence it is easy to see that the classical *Kan formula* holds, providing an equivalence

$$\frac{\mathbf{C} \to \mathscr{X}}{\mathscr{X} \rightleftarrows \mathscr{P}\mathbf{C}} \tag{3}$$

between the category of functors F from an internal $\mathbf{C}$ to a cocomplete $\mathscr{X}$ and that of adjoint pairs $(L \dashv R)$, where arrows are natural transformation between the left adjoints. Observe that R_F is given by

$$R_F(x) = \mathscr{X}(F, x) : \mathbf{C} \to\!\!\!\!\!+ u$$

where x is over u.

2. Change of base

A geometric morphism of topoi $p: \mathscr{F} \to \mathscr{E}$ determines a pair of homomorphisms of bicategories

$$\operatorname{Span} \mathscr{F} \rightleftarrows \operatorname{Span} \mathscr{E}$$

which we will denote by the usual notation p^* and p_*.

We now define the *direct image* $p_*\mathscr{X}$ of a (Span $\mathscr{F}$)-category $\mathscr{X}$. The objects of $p_*\mathscr{X}$ over u are the objects of $\mathscr{X}$ over p^*u, homs are defined by

$$(p_*\mathscr{X})(x, y) = \eta_v^\circ \cdot p_*(\mathscr{X}(x, y)) \cdot \eta_u$$

where x and y are over u and v respectively, and where η's denote the components of the unit of the adjunction $p^* \dashv p_*$. When x, y and z are objects of $p_*\mathscr{X}$ respectively over u, v and w, composition is defined by

$$\begin{aligned} &\eta_w^\circ \cdot p_*\mathscr{X}(y, z) \cdot \eta_v \cdot \eta_v^\circ \cdot p_*\mathscr{X}(x, y) \cdot \eta_u \to \eta_w^\circ \cdot p_*\mathscr{X}(y, z) \cdot p_*\mathscr{X}(x, y) \cdot \eta_u \\ &\cong \eta_w^\circ \cdot p_*(\mathscr{X}(y, z) \cdot \mathscr{X}(x, y)) \cdot \eta_u \to \eta_w^\circ \cdot p_*\mathscr{X}(x, z) \cdot \eta_u . \end{aligned}$$

Here the last arrow is obtained by applying p_* to the composition of $\mathscr{X}$.

Analogously, identities are defined by

$$1 \to \eta_u^\circ \cdot \eta_u \to \eta_u^\circ \cdot p_*(\mathscr{X}(x,x)) \cdot \eta_u$$

where the second arrow is obtained by applying p_* to the identity of $\mathscr{X}$.

Theorem 2.1. *The category $p_*\mathscr{X}$ is locally internal over $\mathscr{E}$, if $\mathscr{X}$ is locally internal over $\mathscr{F}$.*

Proof. In $p_*\mathscr{X}$, substitutions along maps $h: u \to v$ are given by substitutions in $\mathscr{X}$ along the maps $p^*h: p^*u \to p^*v$. □

It is easy to see that the assignment $\mathscr{X} \mapsto p_*\mathscr{X}$ defines a 2-functor

$$p_*: \mathrm{Loc}(\mathscr{F}) \to \mathrm{Loc}(\mathscr{E}).$$

We now define the *inverse image* $p^*\mathscr{Y}$ of a (Span $\mathscr{E}$)-category $\mathscr{Y}$. This is obtained simply by transporting $\mathscr{Y}$ along the homomorphism p^*. An object of $p^*\mathscr{Y}$ is thus an object y of $\mathscr{Y}$, over v, regarded over p^*v. Homs are defined by

$$(p^*\mathscr{Y})(y,z) = p^*(\mathscr{Y}(y,z)).$$

Theorem 2.2 (Change of base). *For any* (Span $\mathscr{F}$)*-category $\mathscr{X}$ and any* (Span $\mathscr{E}$)*-category $\mathscr{Y}$, there is an equivalence*

$$\frac{H: \mathscr{Y} \to p_*\mathscr{X}}{K: p^*\mathscr{Y} \to \mathscr{X}}$$

natural in $\mathscr{X}$ and $\mathscr{Y}$.

Proof. The data for a functor H amount to the effect on objects (Hy is an object of $\mathscr{X}$ over p^*v, for any y over v in $\mathscr{Y}$) and the effect on arrows:

$$\alpha_{y,z}: \mathscr{Y}(y,z) \to \eta_w^\circ \cdot p_*(\mathscr{X}(Hy,Hz)) \cdot \eta_v$$

is a 2-cell in Span $\mathscr{E}$, for any pair of objects y and z, over v and w respectively.

Take $Ky = Hy$ on objects and define the effect of K on arrows by the 2-cell $p^*(\mathscr{Y}(y,z)) \to \mathscr{X}(Hy,Hz)$ obtained by the following pasting:

$$\begin{array}{ccc}
p^*v & \xrightarrow{\;p^*\mathscr{Y}(y,z)\;} & p^*w \\
\downarrow{\scriptstyle p^*\eta_v} & \Downarrow\, p^*\alpha_{yz} & \downarrow{\scriptstyle p^*\eta_w} \\
p^*p_*p^*v & \xrightarrow{\;p^*p_*\mathscr{X}(Hy,Hz)\;} & p^*p_*p^*w \\
\downarrow{\scriptstyle \varepsilon_{p^*v}} & \Downarrow\, \beta_{yz} & \downarrow{\scriptstyle \varepsilon_{p^*w}} \\
p^*v & \xrightarrow{\;\mathscr{X}(Hy,Hz)\;} & p^*w
\end{array}$$

where $\beta_{yz}: p^*p_*\mathscr{X}(Hy, Hz) \to \varepsilon^{\circ}_{p^*w} \cdot \mathscr{X}(Hy, Hz) \cdot \varepsilon_{p^*v}$ is induced by the naturality of ε considered in Span $\mathscr{E}$. Conversely, to assign K means to assign an object Ky of $\mathscr{X}$ over p^*v for every y of $\mathscr{Y}$ over v, and a family of 2-cells

$$\gamma_{yz}: p^*\mathscr{Y}(y,z) \to \mathscr{X}(Ky, Kz)$$

which represents the effect on arrows. Take again $H = K$ on objects, and define the effect of H on arrows by the pasting

$$\begin{array}{ccc}
v & \xrightarrow{\mathscr{Y}(y,z)} & w \\
{\scriptstyle \eta_v}\downarrow & \Downarrow \delta_{yz} & \downarrow{\scriptstyle \eta_w} \\
p_*p^*v & \xrightarrow{p_*p^*\mathscr{Y}(y,z)} & p_*p^*w \\
{\scriptstyle 1}\downarrow & \Downarrow p^*\gamma_{yz} & \downarrow{\scriptstyle 1} \\
p_*p^*v & \xrightarrow{p^*\mathscr{X}(Ky,Kz)} & p_*p^*w
\end{array}$$

where δ_{yz} is induced by the naturality of η in Span $\mathscr{F}$. □

In the following we are interested in the equivalence

$$\frac{\mathbf{C} \to p_*\mathscr{X}}{p^*\mathbf{C} \to \mathscr{X}}$$

only when $\mathscr{X}$ is a locally internal category over $\mathscr{F}$ (and $\mathbf{C}$ is internal to $\mathscr{E}$). Hence, when $\mathbf{D}$ is internal to $\mathscr{F}$ we use $p_*\mathbf{D}$ also to denote the category (internal to $\mathscr{E}$) which is obtained by the image under p_* of the monad defining $\mathbf{D}$. No confusion is possible when $\mathbf{D}$ is regarded as locally internal through the externalization process $\mathscr{L}$ because we have:

Theorem 2.3. $p_*(\mathscr{L}\mathbf{D}) \cong \mathscr{L}(p_*\mathbf{D})$.

Proof. It is easy to check that the bijection

$$\frac{p^*u \to D_0}{u \to p_*D_0}$$

provides objects which correspond under the required isomorphism. □

3. Cocompleteness

Recall that $\mathscr{F}$ can be regarded as the locally internal category $\mathscr{P}1$ over itself. Thus $p_*(\mathscr{P}1)$ can be described as the (Span $\mathscr{E}$)-category whose objects over u are arrows

$1 \rightarrow\!\!\!\!\!\!+\; p^*u$ in Span $\mathscr{F}$, and whose homs are given by

$$p_*(\mathscr{P}1)(\alpha, \beta) = \eta_v^\circ \cdot p_*(\hom_1(\alpha, \beta)) \cdot \eta_u.$$

The assignment $p \mapsto p_*(\mathscr{P}1)$ is the correspondence on objects of a functor

$$(\mathrm{Top}/\mathscr{E})^{\mathrm{co}} \to \mathrm{Loc}(\mathscr{E}).$$

Under this functor, a morphism $h : p \to q$ in Top/$\mathscr{E}$ is taken to the functor $H : p_*(\mathscr{P}1) \to q_*(\mathscr{P}1)$ of (Span $\mathscr{E}$)-categories defined as follows: $H\alpha$ is the composite

$$\tau_u^\circ \cdot h_*\alpha : 1 \rightarrow\!\!\!\!\!\!+\; h_*p^*u \rightarrow\!\!\!\!\!\!+\; q^*u$$

where $\alpha : 1 \rightarrow\!\!\!\!\!\!+\; p^*u$ is an object of $p_*(\mathscr{P}1)$ over u and $\tau : q^* \Rightarrow h_*p^*$ is induced by $p \cong q \cdot h$.

It is easy to verify that $p_*(\mathscr{P}1)$ is a cocomplete category. More generally we have:

Theorem 3.1. *If $\mathscr{X}$ is cocomplete, then also $p_*\mathscr{X}$ is cocomplete.*

Proof. Consider the diagram

$$v \overset{\alpha}{\leftarrow\!\!\!\!\!\!+} I \overset{H}{\longrightarrow} p_*\mathscr{X}$$

where I is an internal category. The colimit $\alpha * H$ is given by the colimit $p^*\alpha * K$ (in $\mathscr{X}$), where K corresponds to H in the change of base. □

The calculus of colimits in $p_*(\mathscr{P}\mathbf{C})$ is particularly easy. Indeed, when $\mathscr{X} = \mathscr{P}\mathbf{C}$, then $\alpha * H : \mathbf{C} \rightarrow\!\!\!\!\!\!+\; p^*v$ is given by the composition

$$\mathbf{C} \overset{H}{\rightarrow\!\!\!\!\!\!+} p^*I \overset{p^*\alpha}{\longrightarrow\!\!\!\!\!\!\!\!+} p^*v.$$

Theorem 3.2. *For any category* $\mathbf{C}$, *internal to* $\mathscr{F}$, *there is an adjunction* $L \dashv R$

$$p_*(\mathscr{P}\mathbf{C}) \underset{L}{\overset{R}{\rightleftarrows}} \mathscr{P}(p_*\mathbf{C}).$$

Proof. Apply the Kan formula (3) to the image under p_* of the Yoneda embedding $\mathbf{C} \to \mathscr{P}\mathbf{C}$, taking into account that $p_*(\mathscr{P}\mathbf{C})$ is cocomplete (Theorem 3.1) and $p_*\mathbf{C}$ is internal. □

For any object $\alpha : \mathbf{C} \rightarrow\!\!\!\!\!\!+\; p^*v$ in $p_*(\mathscr{P}\mathbf{C})$, the module $R(\alpha) : p_*\mathbf{C} \rightarrow\!\!\!\!\!\!+\; v$ is given by the composition

$$p^*\mathbf{C} \overset{p_*\alpha}{\longrightarrow\!\!\!\!\!\!\!\!+} p_*p^*v \overset{\eta_v^\circ}{\rightarrow\!\!\!\!\!\!+} v, \tag{4}$$

and analogously, for any $\beta : p_*\mathbf{C} \nrightarrow u$, the module $L(\beta)$ is given by the composition

$$\mathbf{C} \overset{\varepsilon^\circ_{\mathbf{C}}}{\nrightarrow} p_*p^*\mathbf{C} \overset{p^*\beta}{\nrightarrow} p^*u. \tag{5}$$

where ε denotes the counit of the adjunction $p^* \dashv p_*$.

In particular, taking $\mathbf{C}=\mathbf{1}$ in the previous theorem, we obtain an adjunction

$$p_*(\mathscr{P}\mathbf{1}) \rightleftarrows \mathscr{P}\mathbf{1} \tag{6}$$

which reproduces in terms of (Span $\mathscr{E}$)-categories the original geometric morphism $\mathscr{F} \to \mathscr{E}$. By utilizing the formulas (5) and (4) for the calculation of $L(\beta)$ and $R(\alpha)$ respectively, it is easy to check that, when p is an inclusion, the counit of the adjunction (6) is an isomorphism.

Theorem 3.3. *When* $\mathbf{C}$ *is internal to* $\mathscr{E}$ *and* $\mathbf{D}$ *is internal to* $\mathscr{F}$, *there is an equivalence*

$$\frac{\mathbf{D} \overset{\alpha}{\nrightarrow} p^*\mathbf{C}}{p_*(\mathscr{P}\mathbf{D}) \underset{L}{\overset{R}{\rightleftarrows}} \mathscr{P}\mathbf{C}}$$

between mod($\mathbf{D}, p^*\mathbf{C}$) *and the category of adjoint pairs* ($L \dashv R$), *natural in* $\mathbf{C}$ *and in* $\mathbf{D}$.

Proof. Consider the following sequence of equivalences:

$$\begin{array}{ll} \mathbf{D} \overset{\alpha}{\nrightarrow} p^*\mathbf{C} & \\ \hline & \text{(by the property of } \mathscr{P}\text{)} \\ p^*\mathbf{C} \to \mathscr{P}\mathbf{D} & \\ \hline & \text{(change of base)} \\ \mathbf{C} \to p_*(\mathscr{P}\mathbf{D}) & \\ \hline & \text{(Kan formula)} \\ p_*(\mathscr{P}\mathbf{D}) \underset{L}{\overset{R}{\rightleftarrows}} \mathscr{P}\mathbf{C} & \end{array}$$

□

4. Preservation of limits

Definition 4.1. We say that p^* *preserves limits indexed by the module* $\beta : u \nrightarrow I$ if, for any functor $H : I \to \mathscr{P}\mathbf{C}$, we have

$$p^*\{\beta, H\} \cong \{p^*\beta, p^*H\},$$

when p^*H is regarded as a functor $p^*I \to \mathscr{P}(p^*\mathbf{C})$.

By taking into account the calculus of limits (1), it turns out that p^* preserves limits indexed by β exactly when p^* preserves right liftings along β:

$$p^*(\hom^I(\beta, H)) \cong \hom^{p^*I}(p^*\beta, p^*H) \tag{7}$$

for any module $H: C \nrightarrow I$.

In particular a right lifting of the type $\hom^u(g, \alpha)$, when g is a map, can be calculated as the composition $g^\circ \cdot \alpha$. Hence, p^* preserves limits indexed by maps. Another particular case is when p is an essential morphism. Then p^* preserves all limits and hence it preserves all right liftings.

Let us denote by Φ the class of modules such that limits indexed by the modules of Φ are preserved by p^* in the above sense.

Corollary 4.2. *If p^* preserves the limits indexed by the modules of Φ, then, with the notation of the previous theorem, the following properties are equivalent:*

(i) *the functor $L: \mathscr{P}\mathbf{C} \to p_*(\mathscr{P}\mathbf{D})$ preserves the limit $\{\beta, H\}$ for each β in Φ;*

(ii) *composition with the module α is a functor*

$$- \cdot \alpha : \mathscr{P}(p^*\mathbf{C}) \to \mathscr{P}\mathbf{D}$$

which preserves the limit $\{p^*\beta, p^*H\}$ for each β in Φ.

Proof. (i) is equivalent to

$$\hom^{p^*I}(p^*\beta, p^*H \cdot \alpha) \cong p^* \hom^I(\beta, H) \cdot \alpha$$

for any $H: \mathbf{C} \nrightarrow I$.

(ii) is equivalent to

$$\hom^{p^*I}(p^*\beta, p^*H \cdot \alpha) \cong \hom^{p^*I}(p^*\beta, p^*H) \cdot \alpha.$$

The statement now follows from (7), i.e. from the hypothesis that p^* preserves limits indexed by β. $\square$

As a particular case of the above Corollary 4.2, we obtain in this setting the *Diaconescu theorem* [4]. For this, consider $\mathbf{D} = \mathbf{1}$ and recall that the topoi $\mathscr{E}^{\mathbf{C}^{\mathrm{op}}}$ and $\mathscr{F}^{p^*\mathbf{C}^{\mathrm{op}}}$ are the fibers over 1 of $\mathscr{P}\mathbf{C}$ and $\mathscr{P}(p^*\mathbf{C})$ respectively. Moreover, all finite limits in these categories can be obtained as limits $\{\beta, H\}$, where H has domain in a finite category whose objects are all over 1, and β is a module whose components are all identities. Hence p^* preserves limits indexed by β and Corollary 4.2 applies.

5. Generators

For a category $\mathscr{X}$ enriched in Span $\mathscr{E}$, it is natural to say that the object x_0 over

u is a *generator* whenever, for every pair of objects y and z, the 2-cell

$$\mathscr{X}(y,z) \to \hom^u(\mathscr{X}(x_0,y), \mathscr{X}(x_0,z)) \tag{8}$$

obtained by composition in $\mathscr{X}$, is a monomorphism.

If we denote by $\mathbf{C}$ the internal full subcategory $\mathscr{X}(x_0,x_0)$, this means that, regarding $\mathscr{X}(x_0,y)$ as a module $C \nrightarrow v$, the functor

$$\mathscr{X}(x_0,-) : \mathscr{X} \to \mathscr{P}\mathbf{C}$$

given by $y \mapsto \mathscr{X}(x_0,y)$, is faithful.

Definition 5.1. The object x_0 is said to be a *strong generator* if, for every pair of objects y and z, the 2-cell (8) is the equalizer (in the category of spans $v \nrightarrow w$) of the pair

$$\hom^u(\mathscr{X}(x_0,y), \mathscr{X}(x_0,z)) \rightrightarrows \hom^u(\mathscr{X}(x_0,y), \hom^u(\mathscr{X}(x_0,x_0), \mathscr{X}(x_0,z)))$$

i.e. the functor $\mathscr{X}(x_0,-)$ is fully faithful.

It is now easy to give the link with the usual notion of object of generators. Let us denote again by $\mathbf{C}$ the internal full subcategory of $\mathscr{X}$ generated by the object x_0. Then we have:

Lemma 5.2. *Suppose $\mathscr{X}$ is cocomplete. Then x_0 is a strong generator if and only if the functor $\mathscr{X}(x_0,-) : \mathscr{X} \to \mathscr{P}\mathbf{C}$ has a left adjoint such that the counit is an isomorphism.*

Proof. Apply the Kan formula (3) to the object x_0 regarded as a functor $\mathbf{C} \to \mathscr{X}$. The left adjoint is given by $- * x_0$, and for every y we have $\mathscr{X}(x_0,y) * x_0 \cong y$ because $\mathscr{X}(x_0,-)$ is fully faithful. □

Theorem 5.3. *Suppose p^* preserves the limits indexed by the modules of Φ. Then p is bounded over $\mathscr{E}$ if and only if $p_*(\mathscr{P}\mathbf{1})$ has a strong generator x_0 such that $- * x_0 : \mathscr{P}\mathbf{C} \to p_*(\mathscr{P}\mathbf{1})$ preserves the limits $\{\beta, H\}$, for each β in Φ.*

Proof. The morphism p is bounded over $\mathscr{E}$ if and only if there is an inclusion $\mathscr{F} \to \mathscr{E}^{\mathbf{C}^{\mathrm{op}}}$ over $\mathscr{E}$ [4]. This gives a module $\alpha : 1 \nrightarrow p^*\mathbf{C}$ such that composition with α preserves limits of type $\{p^*\beta, p^*H\}$ with β in Φ. From α we determine a strong generator x_0 of $p_*(\mathscr{P}\mathbf{1})$ and Corollary 4.2 ensures that $- * x : \mathscr{P}\mathbf{C} \to p_*(\mathscr{P}\mathbf{1})$ preserves the limits of type $\{\beta, H\}$. The converse is similar. □

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